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Necessary conditions for metrics in integral Bernstein-type inequalities☆

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Abstract

Let $\mathcal{T}_n$ be the set of all trigonometric polynomials of degree at most n. Denote by Φ^+ the class of all functions φ: $(0, \infty) \to \mathbb{R}$ of the form $\varphi(u) = \psi(\ln u)$, where ψ is nondecreasing and convex on $(-\infty, \infty)$. In 1979, Arestov extended the classical Bernstein inequality $\|T_n'\|_C \le n\|T_n\|_C$, $T_n \in \mathcal{T}_n$, to metrics defined by $\varphi \in \Phi^+$:

$$\int_0^{2\pi} \varphi(|T_n'(t)|)\mathrm{d}t \le \int_0^{2\pi} \varphi(n|T_n(t)|)\mathrm{d}t, \quad T_n \in \mathcal{T}_n.$$

We study the question whether it is possible to extend the class Φ^+, and prove that under certain assumptions Φ^+ is the largest possible class.

Keywords: Algebraic polynomials; Trigonometric polynomials; Bernstein-type inequalities; Integral inequalities

1. Introduction

Let $\mathcal{T}_n$ be the set of all trigonometric polynomials of degree at most n with complex coefficients. The inequality

$$\|T_n'\|_C \le n\|T_n\|_C, \quad T_n \in \mathcal{T}_n, \tag{1}$$

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is well known in approximation theory and is called the Bernstein inequality. Inequality (1) turns into equality iff $T_n(t) = a\cos nt + b\sin nt$, where $a, b \in \mathbb{C}$. The inequality was stated by Bernstein and Landau for polynomials with real coefficients (for details, see [5, Section 10, pp. 25–26; Section 3.4, p. 527], [8, Ch. 6, Theorems 1.2.4, 1.2.5]) in 1912–1914 and by Riesz for polynomials with complex coefficients ([10], [11, Vol. 2, Ch. 10]) in 1914.

We say that a function φ is increasing on an interval I if $\varphi(u_1) \le \varphi(u_2)$ for all $u_1 \le u_2$, $u_1, u_2 \in I$; φ is convex on I if $\varphi\big(\alpha u_1 + (1-\alpha)u_2\big) \le \alpha\varphi(u_1) + (1-\alpha)\varphi(u_2)$ for all $u_1, u_2 \in I$ and $\alpha \in [0, 1]$; φ is concave on I if $-\varphi$ is convex on I.

In 1933, Zygmund [11, Vol. 2, Ch. 10, (3.25)] proved the following statement. If φ is an increasing and convex function on $[0, \infty)$, then

$$\int_0^{2\pi} \varphi\big(|T_n'(t)|\big)\,\mathrm{d}t \le \int_0^{2\pi} \varphi\big(n|T_n(t)|\big)\,\mathrm{d}t, \qquad T_n \in \mathcal{T}_n. \tag{2}$$

For $\varphi(u) = u^p$, $p \ge 1$, inequality (2) implies the Bernstein inequality in the space L_p:

$$\|T_n'\|_p \le n\|T_n\|_p, \qquad T_n \in \mathcal{T}_n,$$

where $\|f\|_p = \left(\frac{1}{2\pi}\int_0^{2\pi} |f(t)|^p\,\mathrm{d}t\right)^{1/p}$.

In 1979, Arestov [1–3] found weaker conditions on functions φ which provide the validity of inequality (2). Before we give Arestov's result, we introduce some notation [2,4].

We denote by Φ^+ the class of functions φ defined on $(0, \infty)$ with the following properties:

(i) φ is locally absolutely continuous;
(ii) φ increases on $(0, \infty)$;
(iii) $u\varphi'(u)$ increases on $(0, \infty)$.

Put $\psi(v) = \varphi(\mathrm{e}^v)$; that is, $\varphi(u) = \psi(\ln u)$. Clearly, φ belongs to Φ^+ iff the function ψ is increasing and convex on $(-\infty, \infty)$. For example, all increasing convex functions, the functions $\ln u$, $\ln^+ u = \max\{0, \ln u\}$, $\ln(1 + u^p)$, and u^p, $p > 0$, belong to Φ^+.

We denote by $\mathcal{P}_n$ the set of all algebraic polynomials of degree at most n with complex coefficients. Let polynomials Λ_n and P_n from $\mathcal{P}_n$ be given by $\Lambda_n(z) = \sum_{k=0}^n \binom{n}{k}\lambda_k z^k$ and $P_n(z) = \sum_{k=0}^n \binom{n}{k} c_k z^k$. The polynomial

$$\Lambda_n P_n(z) = \sum_{k=0}^{n} \binom{n}{k} \lambda_k c_k z^k \tag{3}$$

is called the composition of Λ_n and P_n (for details, see [9, Vol. 2, Section 5]). Suppose that Λ_n is fixed, then Eq. (3) defines a linear operator on $\mathcal{P}_n$, which we denote by the same symbol Λ_n. For example, if $\Lambda_n(z) = (1 + \mathrm{e}^{\mathrm{i}\theta}z)^n$, $\theta \in \mathbb{R}$, then $(\Lambda_n P_n)(z) = P_n(\mathrm{e}^{\mathrm{i}\theta}z)$ is the operator of rotation by angle θ; in particular, $\Lambda_n(z) = (1+z)^n$ defines the identity operator. The polynomial $\Delta_n(z) = \frac{n}{2}(1+z)^{n-1}(z-1)$ defines the differential operator

$$(\Delta_n P_n)(z) = zP_n'(z) - \frac{n}{2}P_n(z).$$

In the sequel, if $P_n \in \mathcal{P}_n$ has degree $m < n$, then we say that $z = \infty$ is a zero of P_n with multiplicity $n - m$. Let $\mathcal{P}_n^0$ be the set of all polynomials $P_n \in \mathcal{P}_n$ such that all n zeros of P_n lie in the unit disk $|z| \le 1$, and let $\mathcal{P}_n^\infty$ be the set of all polynomials $P_n \in \mathcal{P}_n$ such that all zeros of P_n lie in the domain $|z| \ge 1$. Furthermore, we say that an operator Λ_n belongs to the class

Ω_n^0 if $\Lambda_n \mathcal{P}_n^0 \subset \mathcal{P}_n^0$, and that Λ_n belongs to the class Ω_n^∞ if $\Lambda_n \mathcal{P}_n^\infty \subset \mathcal{P}_n^\infty$. Using Theorems 151 and 152 from [9, Section 5] (see also [2]), one can easily prove that $\Lambda_n \in \Omega_n^0$ iff the polynomial $\Lambda_n \in \mathcal{P}_n^0$, and that $\Lambda_n \in \Omega_n^\infty$ iff the polynomial $\Lambda_n \in \mathcal{P}_n^\infty$. Finally, let $\Omega_n = \Omega_n^0 \cup \Omega_n^\infty$.

Theorem A (*Arestov [2]*). *If $\varphi \in \Phi^+$ and $\Lambda_n \in \Omega_n$, then, for all $P_n \in \mathcal{P}_n$,*

$$\int_0^{2\pi} \varphi\big(|\Lambda_n P_n(\mathrm{e}^{\mathrm{i}t})|\big)\,\mathrm{d}t \le \int_0^{2\pi} \varphi\big(C(\Lambda_n)|P_n(\mathrm{e}^{\mathrm{i}t})|\big)\,\mathrm{d}t, \tag{4}$$

where $C(\Lambda_n) = \max\big\{|\lambda_0|, |\lambda_n|\big\}$. Equality holds in (4) if and only if P_n has the form

$$P_n(z) = az^n, \qquad P_n(z) \equiv a, \quad \text{or} \quad P_n(z) = az^n + b \quad (a, b \in \mathbb{C}),$$

depending on whether

$$\Lambda_n \in \Omega_n^0, \qquad \Lambda_n \in \Omega_n^\infty, \quad \text{or} \quad \Lambda_n \in \Omega_n^0 \cap \Omega_n^\infty.$$

The space $\mathcal{T}_n$ can be identified with the space $\mathcal{P}_{2n}$ by the mapping $T_n(t) = \mathrm{e}^{-\mathrm{i}nt} P_{2n}(\mathrm{e}^{\mathrm{i}t})$, $P_{2n} \in \mathcal{P}_{2n}$; moreover,

$$|T_n(t)| = |P_{2n}(\mathrm{e}^{\mathrm{i}t})|, \qquad |T_n'(t)| = |(\Delta_{2n} P_{2n})(\mathrm{e}^{\mathrm{i}t})|.$$

Note that $\Delta_{2n} \in \Omega_{2n}^0 \cap \Omega_{2n}^\infty$ and $C(\Delta_{2n}) = n$. Hence, inequality (2) is a consequence of Theorem A.

Professor Arestov asked the author whether it is possible to extend the class Φ^+ in Theorem A. In this paper we prove that, under certain assumptions, Φ^+ is the largest possible class.

2. Main result

We study inequality (4) for the class $\Phi = \Phi_n$ of functions φ defined on $(0, \infty)$ with the following properties:

(i) φ is continuous on $(0, \infty)$;
(ii) φ increases on $(0, \infty)$;
(iii) for all $P_n \in \mathcal{P}_n$, $\int_0^{2\pi} \varphi\big(|P_n(\mathrm{e}^{\mathrm{i}t})|\big)\,\mathrm{d}t < \infty$.

An example of the function $\varphi(u) = -\exp(1/u)$ shows that the third condition cannot be removed.

Now we will introduce a class $\Phi^- \subset \Phi$ with the property that, for every $\varphi \in \Phi^-$, inequality (4) is not satisfied (as will be stated in Theorem 1).

Definition. Denote by Φ^- the set of all functions $\varphi(u) = \psi(\ln u)$, where $\varphi \in \Phi$, and there exist points $v_1 < v_* < v_2$ and a real number k such that the function

$$\psi(v) - k \cdot v$$

(i) increases on $[v_1, v_*]$ and decreases on $[v_*, v_2]$,
(ii) does not coincide with a constant in any neighborhood of the point v_*.

Remark 1. Let us clarify this definition. Suppose that $\varphi \notin \Phi^+$ and the corresponding function ψ has a locally absolutely continuous derivative ψ' everywhere. Then $\psi''(v_*) < 0$ for some v_*. Hence, there exist points $v_1 < v_* < v_2$ such that

$$\begin{aligned} &\psi'(v) > \psi'(v_*), \quad v \in [v_1, v_*], \\ &\psi'(v) < \psi'(v_*), \quad v \in [v_*, v_2]. \end{aligned} \tag{5}$$

Furthermore, for the function ψ we have the representation

$$\psi(v) - \psi'(v_*)(v - v_*) = \psi(v_*) + \int_{v_*}^{v} \big(\psi'(\eta) - \psi'(v_*)\big)\, d\eta.$$

It follows from (5) that the function $\psi(v) - \psi'(v_*)(v - v_*)$ increases on $[v_1, v_*]$ and decreases on $[v_*, v_2]$. Therefore, φ belongs to Φ^-.

Thus, if $\varphi \in \Phi$ has a locally absolutely continuous derivative on $(0, \infty)$, then either $\varphi \in \Phi^+$ or $\varphi \in \Phi^-$.

Remark 2. If ψ is strictly concave on some interval $[v_1, v_2]$, then $\varphi \in \Phi^-$.

Remark 3. Let us give two examples of functions from Φ^-. For the function $\varphi(u) = u/(1+u)$, by means of which convergence in measure can be defined [6, Ch. 4, Ex. 4.7.60°], the corresponding function $\psi(v) = e^v/(1 + e^v)$ is concave on $[0, \infty)$ and, therefore, $\varphi \in \Phi^-$.

Let $C_0(v)$, $v \in [0, 1]$, be the Cantor function [6, Ch. 3, Prop. 3.6.5], and let $[v]$ denote the integer part of v. The singular function φ defined by $\varphi(e^v) = C_0(v - [v]) + [v]$ also belongs to Φ^-.

Remark 4. It is sufficient to consider only one of the following two cases: $\Lambda_n \in \Omega_n^0$ or $\Lambda_n \in \Omega_n^\infty$. Indeed, applying the methods of de Bruijn and Springer [7] and Arestov [3], consider the map $I = I_n$ on $\mathcal{P}_n$ defined by

$$(IP_n)(z) = z^n P_n(1/z), \quad P_n \in \mathcal{P}_n.$$

It is clear that $|P_n(e^{it})| = |(IP_n)(e^{-it})|$, $t \in [0, 2\pi]$, $P_n \in \mathcal{P}_n$, and

$$\left|(\Lambda_n P_n)(e^{it})\right| = \left|(I(\Lambda_n P_n))(e^{-it})\right| = \left|\big((I\Lambda_n)(IP_n)\big)(e^{-it})\right|, \quad \Lambda_n \in \Omega_n.$$

Moreover, the map I is a bijection of $\mathcal{P}_n^\infty$ onto $\mathcal{P}_n^0$. Therefore, if, say, $\Lambda_n \in \Omega_n^\infty$, then $I\Lambda_n \in \Omega_n^0$. Thus, inequality (7) is valid for an operator Λ_n and a polynomial P_n iff it is valid for $I\Lambda_n$ and IP_n.

The polynomial $\Lambda_n(z) = c(1 + e^{i\theta} z)^n$ defines on $\mathcal{P}_n$ the operator

$$(\Lambda_n P_n)(z) = c P_n(e^{i\theta} z), \quad c \in \mathbb{C}, \theta \in \mathbb{R}. \tag{6}$$

For this operator, inequality (4) turns into equality for every $P_n \in \mathcal{P}_n$, and so operators (6) are excluded from the further consideration.

Theorem 1. *If $\varphi \in \Phi^-$, $\Lambda_n \in \Omega_n$, and Λ_n is not of the form (6), then there exists a polynomial $P_n \in \mathcal{P}_n$ such that*

$$\int_0^{2\pi} \varphi\left(|(\Lambda_n P_n)(e^{it})|\right) dt > \int_0^{2\pi} \varphi\left(C(\Lambda_n)|P_n(e^{it})|\right) dt, \tag{7}$$

where $C(\Lambda_n) = \max\{|\lambda_0|, |\lambda_n|\}$.

Proof. In view of Remark 4, it is sufficient to prove the theorem for

$$\Lambda_n \in \Omega_n^0, \quad \Lambda_n(z) \neq c(1+e^{i\theta}z)^n, \quad c \in \mathbb{C},\ \theta \in \mathbb{R}. \tag{8}$$

Without loss of generality, we can assume that $\lambda_n = 1$. We claim that $|\lambda_0| \le 1$ and $|\lambda_{n-1}| < 1$. Indeed, by conditions (8), Λ_n has n zeros according to multiplicity $z_1, \ldots, z_n$ and all the zeros lie on the unit circle. Consequently,

$$|\lambda_0| = |z_1 \cdots z_n| \le 1, \qquad |\lambda_{n-1}| = \left|\frac{1}{n}(z_1 + \cdots + z_n)\right| \le 1.$$

The last inequality turns into equality only if $z_1 = \cdots = z_n = e^{i\theta}$ for some $\theta \in \mathbb{R}$, but then Λ_n is an operator of the form (6) and we do not consider such operators. Consequently, under our assumptions, $C(\Lambda_n) = \max\{|\lambda_0|, |\lambda_n|\} = 1$, and we must prove that there exists a polynomial $P \in \mathcal{P}_n$ such that

$$\int_0^{2\pi} \varphi\left(|\Lambda_n P(e^{it})|\right) dt - \int_0^{2\pi} \varphi\left(|P(e^{it})|\right) dt > 0. \tag{9}$$

Suppose that $\varphi(u) = \psi(\ln u)$, points $v_1 < v_* < v_2$ and a constant k satisfy conditions (i) and (ii) of the definition of the class Φ^-. Consider the function

$$\widetilde{\varphi}(u) = \varphi(u) - k \cdot \ln u = \psi(\ln u) - k \cdot \ln u,$$

and set $u_1 = e^{v_1}$, $u_2 = e^{v_2}$, $u_* = e^{v_*}$. Clearly, $\widetilde{\varphi}$ increases on $[u_1, u_*]$, decreases on $[u_*, u_2]$, and does not coincide with a constant in any neighborhood of the point u_*.

Let us construct a polynomial $P \in \mathcal{P}_n$ that satisfies (9) in the form

$$P(z) = mz^{n-1}(z-a), \quad a \in (0,1), \quad m > 0.$$

We have $\Lambda_n P(z) = m(z^n - \lambda_{n-1} a z^{n-1}) = mz^{n-1}(z - \lambda_{n-1} a)$,

$$\int_0^{2\pi} \left|\Lambda_n P(e^{it})\right| dt = \int_0^{2\pi} m\left|e^{it} - \lambda_{n-1} a\right| dt = \int_0^{2\pi} m\left|e^{it} - |\lambda_{n-1}| a\right| dt.$$

Let $Q(e^{it}) = m(e^{it} - |\lambda_{n-1}|a)$; then inequality (9) is equivalent to the inequality

$$\int_0^{2\pi} \left[\varphi\left(|Q(e^{it})|\right) - \varphi\left(|P(e^{it})|\right)\right] dt > 0. \tag{10}$$

Let us compare $\left|P(e^{it})\right|^2$ and $\left|Q(e^{it})\right|^2$ on the interval $[0, 2\pi]$. We have

$$\begin{aligned} \left|P(e^{it})\right|^2 &= m^2(1 + a^2 - 2a\cos t), \\ \left|Q(e^{it})\right|^2 &= m^2\left(1 + |\lambda_{n-1}|^2 a^2 - 2|\lambda_{n-1}| a \cos t\right), \end{aligned} \tag{11}$$

and, consequently,

$$\begin{aligned} \frac{1}{m^2}\left(\left|P(e^{it})\right|^2 - \left|Q(e^{it})\right|^2\right) &= 1 + a^2 - 2a\cos t - 1 - |\lambda_{n-1}|^2 a^2 + 2|\lambda_{n-1}| a \cos t \\ &= a\big(1 - |\lambda_{n-1}|\big)\big(a + |\lambda_{n-1}| a - 2\cos t\big). \end{aligned} \tag{12}$$

Let $t_* = \arccos\big((a + |\lambda_{n-1}|a)/2\big)$. Evidently, $t_* \in (0, \pi)$, and it can be verified easily that

$$|P(\mathrm{e}^{\mathrm{i}t_*})| = |Q(\mathrm{e}^{\mathrm{i}t_*})| = m\sqrt{1 - |\lambda_{n-1}|a^2}. \tag{13}$$

It follows from (11) that the absolute values $|P(\mathrm{e}^{\mathrm{i}t})|$ and $|Q(\mathrm{e}^{\mathrm{i}t})|$ are even functions of t that are increasing on $[0, \pi]$; by (12),

$$|Q(\mathrm{e}^{\mathrm{i}t})| > |P(\mathrm{e}^{\mathrm{i}t})|, \quad t \in [0, t_*), \quad \text{and} \quad |Q(\mathrm{e}^{\mathrm{i}t})| < |P(\mathrm{e}^{\mathrm{i}t})|, \quad t \in (t_*, \pi]. \tag{14}$$

Thus, we conclude that the values $\left|Q(\mathrm{e}^{\mathrm{i}t})\right|$ and $\left|P(\mathrm{e}^{\mathrm{i}t})\right|$ belong to the interval $[|P(1)|, |P(-1)|]$ for all $t \in [0, 2\pi]$ and

$$|P(1)| = m(1 - a), \quad |P(-1)| = m(1 + a). \tag{15}$$

Now, we choose parameters m and a such that

$$\left|P(\mathrm{e}^{\mathrm{i}t_*})\right| = \left|Q(\mathrm{e}^{\mathrm{i}t_*})\right| = u_* \quad \text{and} \quad \big[|P(1)|, |P(-1)|\big] \subset [u_1, u_2]. \tag{16}$$

This can be done the following way. Let a_k be a sequence such that $a_k \to +0$, $k \to \infty$. Define m_k by

$$m_k\sqrt{1 - |\lambda_{n-1}|a_k^2} = u_*.$$

Then $m_k \to u_*$, $k \to \infty$. Therefore,

$$m_k(1 - a_k) \to u_* > u_1, \quad \text{and} \quad m_k(1 + a_k) \to u_* < u_2.$$

Thus we can take $a = a_k$ and $m = m_k$ for a sufficiently large value of k.

Combining (14) and (16), we conclude that

$$\begin{aligned} &u_1 \le |P(1)| < |P(\mathrm{e}^{\mathrm{i}t})| < |Q(\mathrm{e}^{\mathrm{i}t})| < u_*, \quad t \in (0, t_*), \\ &u_* < |Q(\mathrm{e}^{\mathrm{i}t})| < |P(\mathrm{e}^{\mathrm{i}t})| < |P(-1)| \le u_2, \quad t \in (t_*, \pi). \end{aligned} \tag{17}$$

It remains to verify inequality (10) for the constructed polynomial P. By the well-known Jensen formula (see, for example, [9, Section 3, Problem 175]),

$$\int_0^{2\pi} \ln|P(\mathrm{e}^{\mathrm{i}t})|\,\mathrm{d}t = \int_0^{2\pi} \ln|m(\mathrm{e}^{\mathrm{i}t} - a)|\,\mathrm{d}t = 2\pi \ln m,$$
$$\int_0^{2\pi} \ln|Q(\mathrm{e}^{\mathrm{i}t})|\,\mathrm{d}t = \int_0^{2\pi} \ln|m(\mathrm{e}^{\mathrm{i}t} - |\lambda_{n-1}|a)| = 2\pi \ln m.$$

Thus,

$$\begin{aligned} &\int_0^{2\pi} \left[\varphi\left(|Q(\mathrm{e}^{\mathrm{i}t})|\right) - \varphi\left(|P(\mathrm{e}^{\mathrm{i}t})|\right)\right]\mathrm{d}t \\ &= \int_0^{2\pi} \left[\varphi\left(|Q(\mathrm{e}^{\mathrm{i}t})|\right) - \varphi\left(|P(\mathrm{e}^{\mathrm{i}t})|\right) - k\ln|Q(\mathrm{e}^{\mathrm{i}t})| + k\ln|P(\mathrm{e}^{\mathrm{i}t})|\right]\mathrm{d}t \\ &= 2\int_0^{\pi} \left[\widetilde{\varphi}\left(|Q(\mathrm{e}^{\mathrm{i}t})|\right) - \widetilde{\varphi}\left(|P(\mathrm{e}^{\mathrm{i}t})|\right)\right]\mathrm{d}t \\ &= 2\int_0^{t_*} \left[\widetilde{\varphi}\left(|Q(\mathrm{e}^{\mathrm{i}t})|\right) - \widetilde{\varphi}\left(|P(\mathrm{e}^{\mathrm{i}t})|\right)\right]\mathrm{d}t + 2\int_{t_*}^{\pi} \left[\widetilde{\varphi}\left(|Q(\mathrm{e}^{\mathrm{i}t})|\right) - \widetilde{\varphi}\left(|P(\mathrm{e}^{\mathrm{i}t})|\right)\right]\mathrm{d}t. \end{aligned}$$

Relations (17) yield that the last expression is greater than 0. This completes the proof of the theorem. □

Corollary 1. *For any $\varphi \in \Phi^-$, there exists $T_n \in \mathcal{T}_n$ such that*

$$\int_0^{2\pi} \varphi\left(|T_n'(t)|\right)\,\mathrm{d}t > \int_0^{2\pi} \varphi\left(n|T_n(t)|\right)\,\mathrm{d}t.$$

For smooth functions $\varphi \in \Phi$, Arestov's theorem and Theorem 1 give the necessary and sufficient conditions on φ for validity of inequality (4).

Corollary 2. *Suppose that an operator $\Lambda_n \in \Omega_n$ is not of the form (6) and a function $\varphi \in \Phi$ has a locally absolutely continuous derivative. Then inequality (4) is valid if and only if $\varphi \in \Phi^+$.*

The proof immediately follows from Theorem A, Remark 1, and Theorem 1.

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References

[1] V.V. Arestov, S.N. Bernšteǐn inequalities for algebraic and trigonometric polynomials, Dokl. Akad. Nauk SSSR 246 (6) (1979) 1289–1292 (in Russian). Translation in Soviet Math. Dokl. 20 (3) (1979) 600–603.
[2] V.V. Arestov, Integral inequalities for trigonometric polynomials and their derivatives, Izv. Akad. Nauk SSSR Ser. Mat. 45 (1) (1981) 3–22 (in Russian). Translation in Math. USSR-Izv. 18 (1982) 1–18.
[3] V.V. Arestov, Integral inequalities for algebraic polynomials on the unit circle, Mat. Zametki 48 (4) (1990) 7–18 (in Russian). Translation in Math. Notes 48 (4) (1990) 977–984.
[4] V.V. Arestov, Inequalities of Bernstein and Szegö for trigonometric polynomials, Izv. Ural. Gos. Univ. Ser. Mat. Mekh. Inform. 58 (2008) 43–58 (in Russian).
[5] S.N. Bernstein, Collected Works. Vol. I. The Constructive Theory of Functions [1905–1930], Izdat. Akad. Nauk SSSR, Moscow, 1952 (in Russian).
[6] V.I. Bogachev, Measure Theory, Vol. I, Springer-Verlag, Berlin, 2007.
[7] N.G. de Bruijn, T.A. Springer, On the zeros of composition-polynomials, Indag. Math. 9 (1947) 406–414.
[8] G.V. Milovanović, D.S. Mitrinović, Th.M. Rassias, Topics in Polynomials: Extremal Problems, Inequalities, Zeros, World Scientific Publishing Co., Inc., River Edge, NJ, 1994.
[9] G. Pólya, G. Szegö, Problems and Theorems in Analysis, Vol. II, Springer-Verlag, New York–Heidelberg, 1976.
[10] M. Riesz, Eine trigonometrische Interpolationsformel und einige Ungleichungen für Polynome, Deutsche Math. Ver. 23 (1914) 354–368.
[11] A. Zygmund, Trigonometric Series, Vol. II, Cambridge University Press, New York, 1959.